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Solving nonlinear complementarity problems by isotonicity of the metric projection ☆

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ABSTRACT

The main motivation for introducing the notion of isotone projection cones was to solve nonlinear complementarity problems. The notion of *-isotone projection cones is introduced by this paper in a similar fashion. Iterative methods for finding solutions of complementarity problems on *-isotone projection cones are presented. The problem of finding nonzero solutions of these problems is also considered.

1. Introduction

Let us call a pointed closed convex cone simply cone. If $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert space, $K \subset H$ a cone, K^* the dual of K, and $f : K \to H$ a mapping, then the nonlinear complementarity problem defined by f and K is the problem of finding an $x^* \in K$ such that $f(x^*) \in K^*$ and $\langle x^*, f(x^*)\rangle = 0$. Complementarity problems are used to model several problems of economics, physics and engineering and they occur in optimization theory too. It is known that x^* is a solution of the nonlinear complementarity problem defined by K and f if and only if x^* is a fixed point of the mapping

$$K \ni x \mapsto P_K\bigl(x - f(x)\bigr), \tag{1}$$

where P_K is the projection mapping onto K. The nonlinear complementarity problem defined by K and f will be denoted by $NCP(f, K)$.

G. Isac and A.B. Németh have characterized a cone in the Euclidean space which admits an isotone projection onto it [1], where isotonicity is considered with respect to the order induced by the cone. They called such a cone isotone projection cone. The same authors [2] and S.J. Bernau [3] considered the similar problem for the Hilbert space. Bearing in mind the fixed point characterization of nonlinear complementarity problems, the isotonicity of the projection provides new existence results and iterative methods [4–7] for these problems. Both the solvability and the approximation of solutions of nonlinear complementarity problems can be handled by using the metric projection onto the cone defining the problem, which emphasize the importance of studying the properties of projection mappings onto cones. In this paper we introduce the notion of *-isotone projection cones. A cone $K \subset H$ is called *-isotone projection cone if $y - x \in K^*$ implies $P_K(y) - P_K(x) \in K$. The main motivation for introducing the *-isotone projection cones was the observation that iterative methods

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for solving the nonlinear complementarity problem similar to those of [7] can be used if the isotone projection cone is replaced by a *-isotone projection cone.

The structure of the paper is as follows. In Section 2 we will fix the terminology and notations, and present some background results used in the paper. In Section 3 we will introduce the *-isotone projection cones and analyze how large is the class of these cones. We will show that each generating *-isotone projection cone is superdual. We will prove that a simplicial cone in $\mathbb{R}^m$ is *-isotone projection cone if and only if it is coisotone. More theoretical results about *-isotone projection cone are in the paper [8] which will be submitted soon. In Section 4 we will consider the solvability of complementarity problems defined by *-isotone projection cones. For this we will use the Picard iteration

$$x^{n+1} = P_K\big(x^n - f\big(x^n\big)\big)$$

for finding the fixed points of the mapping (1). If f is continuous and this iteration is convergent, then its limit is a fixed point of the mapping (1) and therefore a solution of the corresponding fixed point theorem. Moreover, if f is continuous, the sequence $\{x^n\}_{n\in\mathbb{N}}$ is decreasing and the cone K regular, then the limit x^* of $\{x^n\}_{n\in\mathbb{N}}$ is a fixed point of the mapping (1), and therefore a solution of the corresponding complementarity problem. By using the ordering induced by the cone, a sufficient condition is given for f such that $\{x^n\}_{n\in\mathbb{N}}$ to be decreasing. For this we introduced the notion of *-pseudomonotone decreasing mapping and showed that if f is continuous and *-pseudomonotone decreasing, then $\{x^n\}_{n\in\mathbb{N}}$ is decreasing. The *-pseudomonotone decreasing mapping is a mapping which satisfies the following implication:

$$y - x \in K \quad \text{and} \quad f(y) \in K^* \quad \text{implies } f(x) \in K^*.$$

This class of mappings extends the set of mappings which satisfy the following isotonicity property:

$$y - x \in K \quad \Longrightarrow \quad f(x) - f(y) \in K^*.$$

By introducing other types of isotonicity properties for f, we will also analyze the problem of finding nonzero solutions for the complementarity problem.

2. Preliminaries

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space. All Hilbert spaces in this paper are assumed to be real Hilbert spaces. For the simplicity of the terminology from now on we shall call a pointed closed convex cone simply cone. Thus, a closed subset $K \subset H$ is called a *cone* if

(i) $\lambda x \in K$, for all $x \in K$ and $\lambda > 0$ and if
(ii) $x + y \in K$, for all $x, y \in K$.
(iii) $K \cap (-K) = \{0\}$.

$K - K$ is called the *linear subspace generated by* K and it is the smallest linear subspace of H containing K. A cone $K \subset H$ is called *generating* if the linear subspace generated by K is H, that is, $K - K = H$. If $K \subset H$ is a cone, then

$$K^* = \big\{y \in H\colon \langle x, y\rangle \geqslant 0 \text{ for all } x \in K\big\}$$

is called the *dual* of K. A cone $K \subset H$ is called *superdual* if $K^* \subset K$. If K is a cone, then

$$K^\perp = \big\{x \in H\colon \langle x, y\rangle \leqslant 0,\ \forall y \in K\big\}$$

is called the *polar* of K. It is easy to see that $K^\perp = -K^*$. We say that the set A is *generating the cone* K if

$$K = \big\{\lambda_1 x^1 + \cdots + \lambda_\ell x^\ell\colon \ \ell \in \mathbb{N},\ \lambda_1, \ldots, \lambda_\ell \geqslant 0 \text{ and } x^1, \ldots, x^\ell \in A\big\}.$$

Note that if K is a generating cone, then K^* and $K^\perp$ are cones. This is the case for example if K is a *simplicial cone* in $\mathbb{R}^m$, that is, a cone generated by m linearly independent vectors.

The generating cones K and L are called *mutually polar* if $K = L^\perp$ (or equivalently $K^\perp = L$).

A relation ρ on H is called *reflexive* if $x\rho x$ for all $x \in H$. A relation ρ on H is called *transitive* if $x\rho y$ and $y\rho z$ imply $x\rho z$. A relation ρ on H is called *antisymmetric* if $x\rho y$ and $y\rho x$ imply $x = y$. A relation ρ on H is called an *order* if it is reflexive, transitive and antisymmetric. A relation ρ on H is called *translation invariant* if $x\rho y$ implies $(x+z)\rho(y+z)$ for any $z \in H$. A relation ρ on H is called *scale invariant* if $x\rho y$ implies $(\lambda x)\rho(\lambda y)$ for any $\lambda > 0$. A relation ρ on H is called *continuous* if for any two convergent sequences $\{x^n\}_{n\in\mathbb{N}}$ and $\{y^n\}_{n\in\mathbb{N}}$ with $x^n\rho y^n$ for all $n \in \mathbb{N}$ we have $x^*\rho y^*$, where x^* and y^* are the limits of $\{x^n\}_{n\in\mathbb{N}}$ and $\{y^n\}_{n\in\mathbb{N}}$, respectively.

The relation ρ on H is a continuous, translation and scale invariant order if and only if it is induced by a cone $K \subset H$, that is, $\rho = \leqslant_K$, where $x \leqslant_K y$ if and only if $y - x \in K$. The cone K can be written as $K = \{x \in H\colon\ 0 \leqslant_K x\}$ and it is called the positive cone of the order $\leqslant_K$. The triplet $(H, \langle\cdot,\cdot\rangle, K)$ is called an *ordered vector space*. A cone $K \subset H$ is called *regular* if every decreasing sequence of elements in K is convergent. In $\mathbb{R}^m$ any cone is regular. The ordered vector space $(H, \langle\cdot,\cdot\rangle, K)$

is called a *vector lattice* if for every $x, y \in H$ there exist $x \wedge y := \inf\{x, y\}$ and $x \vee y := \sup\{x, y\}$. In this case we say that the cone K is *latticial* and for each $x \in H$ we denote $x^+ = 0 \vee x$, $x^- = 0 \vee (-x)$ and $|x| = x \vee (-x)$. Then, $x = x^+ - x^-$ and $|x| = x^+ + x^-$. If $H = \mathbb{R}^m$, then the latticial cones are exactly the simplicial cones.

Let $P_K : H \to H$ be the *projection mapping* onto the cone K defined by $P_K(x) \in K$ and $\|x - P_K(x)\| = \min\{\|x - y\|:\ y \in K\}$. By using the definition of the metric projection and item (i) of the definition of a cone, it is also easy to show that $P_K(\lambda x) = \lambda P_K(x)$, for any $x \in H$ and any $\lambda \geqslant 0$.

The following theorem is proved in [9].

Theorem 1 *(Moreau). Let H be a Hilbert space and $K, L \subset H$ two mutually polar generating cones in H. Then, the following statements are equivalent*:

(i) $z = x + y$, $x \in K$, $y \in L$ *and* $\langle x, y\rangle = 0$,
(ii) $x = P_K(z)$ *and* $y = P_L(z)$.

If K is a cone, then it is called an *isotone projection cone* if

$$x \leqslant_K y \quad \Longrightarrow \quad P_K(x) \leqslant_K P_K(y).$$

If K is a generating isotone projection cone, then it is latticial [2]. K is called *coisotone cone* if $K^\perp$ is a generating isotone projection cone [10]. If $H = \mathbb{R}^m$, the coisotone cones are exactly the simplicial cones generated by m linearly independent vectors which form pairwise nonacute angles [1,11].

If $K \subset H$ is a cone, K^* the dual cone of K, and $f : K \to H$ a mapping, then the nonlinear complementarity problem defined by f and K is the problem of finding an $x^* \in K$ such that $f(x^*) \in K^*$ and $\langle x^*, f(x^*)\rangle = 0$.

3. *-Isotone projection cones in Hilbert spaces

Definition 1. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $K, L \subset H$ be cones. The mapping $\zeta : H \to H$ is called (L, K)*-isotone* if $x \leqslant_L y$ implies $\zeta(x) \leqslant_K \zeta(y)$. If $L = K$, then ζ is called L*-isotone*.

Definition 2. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $K \subset H$ be a cone. If $P_K : H \to H$ is (K^*, K)-isotone, then the cone K is called **-isotone projection cone*.

The following proposition shows that the class of generating *-isotone projection cones is contained in the class of generating superdual cones.

Proposition 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $K \subset H$ be a generating cone. If the cone K is *-isotone projection cone, then it is superdual.*

Proof. Let x be an arbitrary element of K^*. Since K is generating, there exist $u, v \in K$ such that $x = v - u$. Then, $u \leqslant_{K^*} v$. Since K is *-isotone projection cone, it follows that $u = P_K(u) \leqslant_K P_K(v) = v$ and consequently $x \in K$. Thus, $K^* \subset K$, that is, K is superdual. □

Theorem 2. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $K \subset H$ be a cone. The cone K is *-isotone projection cone, if and only if $P_K(u+v) \leqslant_K u$ for any $u \in K$ and any $v \in K^\perp$.*

Proof. Suppose that K is an *-isotone projection cone. Let $u \in K$ and $v \in K^\perp$ be arbitrary. Then, $u + v \leqslant_{K^*} u$ implies that

$$P_K(u+v) \leqslant_K P_K(u) = u.$$

Conversely, suppose that

$$P_K(u+v) \leqslant_K u \tag{2}$$

for any $u \in K$ and any $v \in K^\perp$. Let $x, y \in H$ with $x \leqslant_{K^*} y$. Then, by Moreau's theorem $x \leqslant_{K^*} y \leqslant_{K^*} P_K(y)$. Thus,

$$x \leqslant_{K^*} P_K(y). \tag{3}$$

Let $u = P_K(y)$ and $v = x - P_K(y)$. Then, obviously $u \in K$ and, by Eq. (3), $v \in K^\perp$. Hence, we can use Eq. (2) to obtain

$$P_K(x) = P_K(u+v) \leqslant_K u = P_K(y).$$

Therefore, K is *-isotone projection cone. □

Corollary 1. *Let $(H, \langle \cdot,\cdot \rangle)$ be a Hilbert space and $K \subset H$ be a cone. The cone K is *-isotone projection cone, if and only if $P_K(x) \leqslant_K u$ for any $u \in K$ and any $x \in H$ with $x \leqslant_{K^*} u$.*

Proof. Use Theorem 2 with $v = x - u \in K^\perp$. $\square$

4. *-Isotone projection cones in Euclidean spaces

Let $\mathbb{R}^m$ be endowed with a Cartesian reference system. All matrices considered in this paper will have real entries. We identify all vectors in $\mathbb{R}^m$ by column vectors. We denote the components of a vector in $\mathbb{R}^m$ by putting lower indices to the letter which denotes the vector. In $\mathbb{R}^m$ the simplicial cones are the cones generated by m linearly independent vectors, that is, the cones of the form $L = A\mathbb{R}^m_+$, where A is a nonsingular $(m \times m)$-matrix. The generators of the cone L are the column vectors of A. Let $\langle \cdot,\cdot \rangle$ be the canonical scalar product of $\mathbb{R}^m$.

Lemma 1. *Let $K \subset \mathbb{R}^m$ be a cone and A a nonsingular matrix. Then,*

$$(AK)^* = \left(A^\top\right)^{-1} K^* \tag{4}$$

and

$$(AK)^\perp = \left(A^\top\right)^{-1} K^\perp. \tag{5}$$

In particular,

$$\left(A\mathbb{R}^m_+\right)^* = \left(A^\top\right)^{-1} \mathbb{R}^m_+ \tag{6}$$

and

$$\left(A\mathbb{R}^m_+\right)^\perp = -\left(A^\top\right)^{-1} \mathbb{R}^m_+. \tag{7}$$

Proof. Eq. (5) follows easily from Eq. (4). Thus, it is enough to prove Eq. (4) only. $x \in (AK)^*$ if and only if $\langle x, Au \rangle \geqslant 0$ for any $u \in K$, which is equivalent to $\langle A^\top x, u \rangle \geqslant 0$ for any $u \in K$, or to $A^\top x \in K^*$. Hence, $x \in (AK)^*$ if and only if $x \in (A^\top)^{-1} K^*$. Therefore, $(AK)^* = (A^\top)^{-1} K^*$. $\square$

For a vector $x \in \mathbb{R}^m$ denote $x^+ = \sup(x, 0)$ and $x^- = \sup(-x, 0)$, where the supremums are taken with respect to the order induced by $\mathbb{R}^m_+$. We write $x \geqslant 0$ if all components of x are nonnegative.

The next proposition is a straightforward application of Moreau's theorem. However, for the readers' convenience we present all details of this proof.

Proposition 2. *Let A be a nonsingular matrix and $K = A\mathbb{R}^m_+$ the corresponding simplicial cone. Then, for any $y \in \mathbb{R}^m$ there exists a unique $x \in \mathbb{R}^m$ such that one of the following two equivalent statements hold*:

(i) $y = Ax^+ - (A^\top)^{-1} x^-$, $x \in \mathbb{R}^m$,
(ii) $Ax^+ = P_K(y)$ *and* $-(A^\top)^{-1} x^- = P_{K^\perp}(y)$.

Proof. Let us first prove that the statements (i) and (ii) are equivalent.

Suppose that (i) holds. Then, by using Eq. (7), it follows that $Ax^+ \in K$, $-(A^\top)^{-1} x^- \in K^\perp$ and

$$\left\langle Ax^+, -\left(A^\top\right)^{-1} x^- \right\rangle = -\left\langle Ax^+, \left(A^{-1}\right)^\top x^- \right\rangle = -\left\langle x^+, x^- \right\rangle = 0.$$

Thus, (ii) follows from Moreau's theorem. The converse follows easily from the same theorem.

Next, we show that there exists a unique x such that item (i) holds. From Moreau's theorem, there exist a unique $p \in K$ and a unique $q \in K^\perp$ such that $y = p + q$ and $\langle p, q \rangle = 0$. Since $K = A\mathbb{R}^m_+$ and A is nonsingular, there exists a unique $u \in \mathbb{R}^m_+$ such that $p = Au$. By using Eq. (7), $K^\perp = -(A^\top)^{-1} \mathbb{R}^m_+$, and hence there exists a unique $v \in \mathbb{R}^m_+$ such that $q = -(A^\top)^{-1} v$. Thus,

$$0 = \langle p, q \rangle = \left\langle Au, -\left(A^\top\right)^{-1} v \right\rangle = -\left\langle Au, \left(A^{-1}\right)^\top v \right\rangle = -\langle u, v \rangle.$$

Therefore, $\langle u, v \rangle = 0$. Let $x = u - v$. Then, $u \in \mathbb{R}^m_+$, $v \in \mathbb{R}^m_+$ and $\langle u, v \rangle = 0$ implies $u = x^+$ and $v = x^-$. In conclusion,

$$y = p + q = Au - \left(A^\top\right)^{-1} v = Ax^+ - \left(A^\top\right)^{-1} x^-. \qquad \square$$

Recall that a square matrix is called *positive stable* if all its eigenvalues have positive real part. A real square matrix is called a *Z-matrix* if all of its off-diagonal entries are nonpositive. An *M-matrix* is a Z-matrix whose eigenvalues are positive.

Therefore, a symmetric matrix is an M-matrix if and only if it is a positive semidefinite Z-matrix. There are a large number of papers and books dealing with the properties and applications of the above classes of matrices. The detailed presentation of this topic is beyond the scope of this paper. However, the reader can find more details about the special classes of M- and Z-matrices in [12–14]. A *Stieltjes matrix* is a symmetric positive definite Z-matrix [15]. It is easy to see that a symmetric matrix is an M-matrix if and only if it is a Stieltjes matrix. It is known that a Z-matrix A is an M-matrix if and only if $Av \geqslant 0$ implies $v \geqslant 0$ [12,13]. All square matrices A satisfying the property "$Av \geqslant 0$ implies $v \geqslant 0$" are called *inverse positive* or *monotone* [12]. Hence, all M-matrices (and in particular the Stieltjes matrices) are inverse positive. However, the converse of this statement is not true. It is known that a square matrix A is inverse positive if and only if $A^{-1} \geqslant 0$ [12].

Proposition 3. *An $m \times m$ positive definite matrix B is a Stieltjes matrix if and only if*

$$u \geqslant 0 \quad \text{and} \quad x^- + B\big(u - x^+\big) \geqslant 0 \quad \text{implies } u - x^+ \geqslant 0. \tag{8}$$

Proof. First we show that any Stieltjes matrix B satisfies (8). We prove this by using induction on the dimension of the matrix. If B_1 is a one-dimensional Stieltjes matrix, then $B_1 = (a)$, where $a > 0$. Let $u, x \in \mathbb{R}$. Thus, we have to show that $u \geqslant 0$ and $x^- + a(u - x^+) = x^- + B_1(u - x^+) \geqslant 0$ implies $u - x^+ \geqslant 0$. If $x \leqslant 0$ this is trivial because $u - x^+ = u \geqslant 0$. If $x \geqslant 0$, then $0 \leqslant x^- + a(u - x^+) = a(u - x^+)$ and hence $u - x^+ \geqslant 0$. Suppose, that the statement is true for m and prove it for $m+1$. We have to show that $\begin{pmatrix} u_1 \\ \vdots \\ u_{m+1} \end{pmatrix} \geqslant 0$ and

$$\begin{pmatrix} x_1^- \\ \vdots \\ x_{m+1}^- \end{pmatrix} + B_{m+1} \begin{pmatrix} u_1 - x_1^+ \\ \vdots \\ u_{m+1} - x_{m+1}^+ \end{pmatrix} \geqslant 0 \tag{9}$$

implies $\begin{pmatrix} u_1 - x_1^+ \\ \vdots \\ u_{m+1} - x_{m+1}^+ \end{pmatrix} \geqslant 0$, where B_{m+1} is an $m+1$-dimensional Stieltjes matrix.

If all components of $\begin{pmatrix} x_1 \\ \vdots \\ x_{m+1} \end{pmatrix}$ are nonnegative, then inequality (9) becomes

$$B_{m+1} \begin{pmatrix} u_1 - x_1^+ \\ \vdots \\ u_{m+1} - x_{m+1}^+ \end{pmatrix} \geqslant 0. \tag{10}$$

Since B_{m+1} is a Stieltjes matrix, it is also an M-matrix and hence inverse positive. Thus, $\begin{pmatrix} u_1 - x_1^+ \\ \vdots \\ u_{m+1} - x_{m+1}^+ \end{pmatrix} \geqslant 0$. Hence, we can suppose that at least one component of $\begin{pmatrix} x_1 \\ \vdots \\ x_{m+1} \end{pmatrix}$ is negative. Thus, there exists $k \in \{1, \dots, m+1\}$ such that $x_k < 0$. Denote by Π the permutation matrix obtained by swapping the k-th line and the $(m+1)$-th line of the $(m+1) \times (m+1)$ identity matrix. Then inequality (9) becomes

$$\Pi \begin{pmatrix} x^- \\ x_k^- \end{pmatrix} + B_{m+1} \Pi \begin{pmatrix} u - x^+ \\ u_k - x_k^+ \end{pmatrix} \geqslant 0, \tag{11}$$

where x^- and $u - x^+$ are given by the equations $\Pi \begin{pmatrix} x^- \\ x_k^- \end{pmatrix} = \begin{pmatrix} x_1^- \\ \vdots \\ x_{m+1}^- \end{pmatrix}$ and $\Pi \begin{pmatrix} u - x^+ \\ u - x_k^+ \end{pmatrix} = \begin{pmatrix} u - x_1^+ \\ \vdots \\ u - x_{m+1}^+ \end{pmatrix}$. Denote by I_{m+1} the $(m+1) \times (m+1)$ identity matrix. Now, we multiply (11) by Π and use $\Pi^2 = I_{m+1}$ to obtain

$$\begin{pmatrix} x^- \\ x_k^- \end{pmatrix} + \Pi B_{m+1} \Pi \begin{pmatrix} u - x^+ \\ u_k - x_k^+ \end{pmatrix} \geqslant 0. \tag{12}$$

Since $\Pi^\top = \Pi$ and B_{m+1} is positive definite, it follows that $\Pi B_{m+1} \Pi$ is also positive definite. Moreover, since B_{m+1} is a Z-matrix, it follows easily that $\Pi B_{m+1} \Pi$ is also a Z-matrix. Thus, $\Pi B_{m+1} \Pi$ is a Stieltjes matrix and hence it can be written in the form

$$\Pi B_{m+1} \Pi = \begin{pmatrix} B_m & -b \\ -b^\top & c \end{pmatrix},$$

where B_m is an m-dimensional Stieltjes matrix, b is an $m \times 1$ nonnegative column vector and c is a positive number. Hence, inequality (12) becomes

$$\begin{pmatrix} x^- \\ x_k^- \end{pmatrix} + \begin{pmatrix} B_m & -b \\ -b^\top & c \end{pmatrix} \begin{pmatrix} u - x^+ \\ u_k - x_k^+ \end{pmatrix} \geqslant 0. \tag{13}$$

Thus, from inequality (13) and $x_k^+ = 0$, it follows that $x^- + B_m(u - x^+) - bu_k \geqslant 0$ which implies

$$x^- + B_m\bigl(u - x^+\bigr) \geqslant bu_k \geqslant 0. \tag{14}$$

Since, B_m is an m-dimensional Stieltjes matrix and our statement is true for m, it follows that B_m satisfies (8). Thus, inequality (14) implies that $u - x^+ \geqslant 0$. Therefore, $\begin{pmatrix} u-x_1^+ \\ \vdots \\ u-x_{m+1}^+ \end{pmatrix} = \Pi \begin{pmatrix} u-x^+ \\ u_k-x_k^+ \end{pmatrix} = \Pi \begin{pmatrix} u-x^+ \\ u_k \end{pmatrix} \geqslant 0.$

Next, we show that if an $m \times m$ positive definite matrix $B = (b_{ij})_{1 \leqslant i,j \leqslant m}$ satisfies (8), then it is a Stieltjes matrix. Suppose to the contrary, that B is an $m \times m$ positive definite matrix which satisfies (8), but it is not a Stieltjes matrix. Then, there exist $i, j \in \{1, \dots, m\}$ such that $i \neq j$ and $b_{ij} > 0$. Choose the vectors $x, u \in \mathbb{R}^m$ such that $u_k = 0$ for all $k \neq j$, $x_i > 0$ and u_j, $-x_k$ are positive and large enough to have

$$b_{ij}u_j - b_{ii}x_i \geqslant 0 \tag{15}$$

and

$$-x_k + b_{kj}u_j - b_{ki}x_i \geqslant 0; \quad k \neq i. \tag{16}$$

Then, $u \geqslant 0$ and

$$x^- + B\bigl(u - x^+\bigr) \geqslant 0 \tag{17}$$

because (15) is the i-th line of (17) and (16) is the k-th line of (17) for any $k \neq i$. On the other hand $u_i - x_i^+ = -x_i < 0$. Thus (8) cannot hold. This contradiction shows that B must be a Stieltjes matrix. □

Theorem 3. *Let A be an $m \times m$ nonsingular matrix and $K = A\mathbb{R}^m_+$ a simplicial cone. Then, K is a *-isotone projection cone if and only if $A^\top A$ is a Stieltjes matrix.*

Proof. First, note that $A^\top A$ is positive definite. Hence, the condition "$A^\top A$ is a Stieltjes matrix" makes sense.

By Corollary 1, K is a *-isotone projection cone if and only if "$y \in \mathbb{R}^m$ and $Au \in K$ with $u \in \mathbb{R}^m_+$ such that $y \leqslant_{K^*} Au$" implies "$P_K(y) \leqslant_K Au$". By Proposition 2, the relation $P_K(y) \leqslant_K Au$ is equivalent to $Au - Ax^+ \in A\mathbb{R}^m_+$, or to $u - x^+ \geqslant 0$, where x is the element uniquely determined by the equation in the item (i) of Proposition 2. On the other hand, by Eq. (6), the relation $y \leqslant_{K^*} Au$ is equivalent to

$$A\bigl(u - x^+\bigr) + \bigl(A^\top\bigr)^{-1}x^- = Au - Ax^+ + \bigl(A^\top\bigr)^{-1}x^- = Au - y \in K^* = \bigl(A^\top\bigr)^{-1}\mathbb{R}^m_+,$$

or to, $x^- + A^\top A(u - x^+) \geqslant 0$. Thus, K is a *-isotone projection cone if and only if $x^- + A^\top A(u - x^+) \geqslant 0$ implies $u - x^+ \geqslant 0$. Therefore, by Proposition 3, K is a *-isotone projection cone if and only if $A^\top A$ is a Stieltjes matrix. □

In $\mathbb{R}^m$ the coisotone cones are simplicial cones with the generators forming pairwise nonacute angles. This means that the coisotone cones in $\mathbb{R}^m$ are the simplicial cones of the form $A\mathbb{R}^m_+$, where A is a nonsingular $(m \times m)$-matrix such that $A^\top A$ is a Stieltjes matrix.

Hence, we have the following theorem which shows that in $\mathbb{R}^m$ the class of coisotone cones is equal to the class of simplicial *-isotone projection cones.

Theorem 4. *Let A be an $m \times m$ nonsingular matrix and $K = A\mathbb{R}^m_+$ a simplicial cone. Then, K is a *-isotone projection cone if and only if it is coisotone.*

Proof. K is coisotone if and only if $A^\top A$ is a Stieltjes matrix. Therefore, the result follows from Theorem 3. □

5. Nonlinear complementarity problems on *-isotone projection cones

Let K be a *-isotone projection cone and $f : K \to H$ be a continuous map. We consider the recursion

$$x^{n+1} = P_K\bigl(x^n - f\bigl(x^n\bigr)\bigr), \tag{18}$$

where n is a nonnegative integer and $x^0 \in K$. If the sequence $\{x^n\}_{n\in\mathbb{N}}$ is convergent to $x^* \in K$ and the mapping f is continuous, then taking the limit in the recursion (18) as m approaches infinity, we obtain that x^* is a fixed point of the mapping $K \ni x \mapsto P_K(x - f(x))$ and therefore a solution of the nonlinear complementarity problem defined by K and f.

We will also consider recursion (18) with f replaced by a positive scaling of f. As a particular case, we will consider the problem of finding the zeros of f. Recursions for complementarity problems, variational inequalities and optimization problems, similar to (18), were considered in several other works, for example [16–26]. However, neither of these works used the order induced by the cone for analyzing the convergence. Instead, they used the Banach fixed point theorem based approach, assuming the Kachurovskii–Minty–Browder type monotonicity (see [27–30]) and global Lipschitz properties for f.

First we state two lemmas from [7] on which our main results are based.

Lemma 2. *Let H be a Hilbert space, $K \subset H$ a cone and $f : K \to H$ a continuous mapping. Consider the recursion* (18)*. If the sequence $\{x^n\}_{n\in\mathbb{N}}$ is convergent and x^* is its limit, then x^* is a solution of the nonlinear complementarity problem NCP(f, K).*

Lemma 3. *Let H be a Hilbert space, $K \subset H$ a regular cone and $f : K \to H$ a continuous mapping. Consider the recursion* (18)*. If the sequence $\{x^n\}_{n\in\mathbb{N}}$ is monotone decreasing, then it is convergent and its limit x^* is a solution of the nonlinear complementarity problem NCP(f, K).*

Definition 3. Let H be a Hilbert space, $K \subset H$ a cone. The mapping $f : K \to H$ is called a **-increasing* if f is (K, K^*)-isotone. The mapping f is called **-decreasing* if $-f$ is *-increasing.

The following notion is inspired by the notion of pseudomonotonicity defined by Karamardian and Schaible in [31].

Definition 4. Let H be a Hilbert space, $K \subset H$ a cone. The mapping $f : K \to H$ is called a **-pseudomonotone decreasing* if for every $x, y \in K$

$$y - x \in K \quad \text{and} \quad f(y) \in K^* \quad \text{implies } f(x) \in K^*.$$

Remark 1.

1. If f is *-decreasing, then it is *-pseudomonotone decreasing.
2. If $f(K) \subset K^*$, then f is *-pseudomonotone decreasing.

Theorem 5. *Let H be a Hilbert space, $K \subset H$ a regular *-isotone projection cone and $f : K \to H$ a continuous mapping such that $f^{-1}(K^*) \neq \emptyset$. Consider the recursion* (18) *starting from an $x^0 \in f^{-1}(K^*)$. If f is *-pseudomonotone decreasing, then the sequence $\{x^n\}_{n\in\mathbb{N}}$ is convergent and its limit x^* is a solution of the nonlinear complementarity problem NCP(f, K).*

Proof. Given that *-isotone projection cone is regular, by Lemma 3, it is enough to prove that the sequence $\{x^n\}_{n\in\mathbb{N}}$ is monotone decreasing. Moreover, it is enough to prove that $f(x^n) \in K^*$ for all $n \in \mathbb{N}$. Indeed, since K is a *-isotone projection cone, $f(x^n) \in K^*$ and $x^n \in K$ imply that

$$x^{n+1} = P_K\bigl(x^n - f\bigl(x^n\bigr)\bigr) \leqslant_K P_K\bigl(x^n\bigr) = x^n. \tag{19}$$

Hence, the sequence $\{x^n\}_{n\in\mathbb{N}}$ is monotone decreasing. We will prove the proposition

$$(\Gamma_n) \quad f\bigl(x^n\bigr) \in K^* \quad \forall n \in \mathbb{N}$$

by induction. (Γ_0) is obviously true. We suppose that (Γ_n) is true and prove that (Γ_{n+1}) is also true. Since $f(x^n) \in K^*$, by relation (19) we have that $x^{n+1} \leqslant_K x^n$. Since f is *-pseudomonotone decreasing we have $f(x^{n+1}) \in K^*$; that is, (Γ_{n+1}) is true. □

Example 1. The monotone nonnegative cone in $\mathbb{R}^m$ is defined in Example 2.13.9.4 of [32] at p. 198. The monotone nonnegative cone is an important cone in the isotonic regression and its applications (see [33] and the references therein). The monotone nonnegative is also used in reconstruction problems (see [32, Section 5.13 and Remark 5.13.2.4]). Suppose that K is the dual of the monotone nonnegative cone in $\mathbb{R}^3$. Then, by Eqs. (435) and (429) of [32] we have

$$K = \bigl\{x \in \mathbb{R}^3 \bigm| x_1 + x_2 + x_3 \geqslant 0,\ x_1 + x_2 \geqslant 0,\ x_1 \geqslant 0\bigr\} \tag{20}$$

and

$$K^* = \bigl\{x \in \mathbb{R}^3 \bigm| x_1 \geqslant x_2 \geqslant x_3 \geqslant 0\bigr\}.$$

It is a straightforward exercise to check that $K^* = U\mathbb{R}^3_+$, where $U = \begin{pmatrix} 1&1&1\\ 0&1&1\\ 0&0&1 \end{pmatrix}$. Then, $(U^\top)^{-1} = \begin{pmatrix} 1&0&0\\ -1&1&0\\ 0&-1&1 \end{pmatrix}$ and therefore, by using Eq. (6), we get $K = (K^*)^* = \begin{pmatrix} 1&0&0\\ -1&1&0\\ 0&-1&1 \end{pmatrix}\mathbb{R}^3_+$. The generators of K are the column vectors of $\begin{pmatrix} 1&0&0\\ -1&1&0\\ 0&-1&1 \end{pmatrix}$ which form pairwise nonacute angles. Thus, K is a coisotone cone, which by Theorem 4 is a *-isotone projection cone. Consider the mapping $f = (f_1, f_2, f_3)^\top : K \to H$ defined by

$$f(x) = \left(3000 - x_1^3 - 2x_1 - 2x_2 - x_3, 2000 - x_1^3 - x_1 - x_2, 1000 - x_1^3\right)^\top. \tag{21}$$

We will show that f is *-pseudomonotone decreasing. For this we have to show that for every $x, y \in K$

$$y - x \in K \quad \text{and} \quad f(y) \in K^* \quad \text{implies } f(x) \in K^*,$$

or equivalently

$$\begin{cases} y_1 + y_2 + y_3 \geqslant x_1 + x_2 + x_3, \\ y_1 + y_2 \geqslant x_1 + x_2, \\ y_1 \geqslant x_1 \end{cases} \tag{22}$$

and

$$f_1(y) \geqslant f_2(y) \geqslant f_3(y) \geqslant 0 \tag{23}$$

implies

$$f_1(x) \geqslant f_2(x) \geqslant f_3(x) \geqslant 0. \tag{24}$$

Obviously, inequalities $(22)_3$ and $(23)_3$ imply inequality $(24)_3$. It is easy to see that

$$f_2(y) - f_3(y) = 1000 - y_1 - y_2$$

and

$$f_2(x) - f_3(x) = 1000 - x_2 - x_3.$$

Therefore, inequalities $(22)_2$ and $(23)_2$ imply inequality $(24)_2$. It is straightforward to see that

$$f_1(y) - f_2(y) = 1000 - y_1 - y_2 - y_3$$

and

$$f_1(x) - f_2(x) = 1000 - x_1 - x_2 - x_3.$$

Therefore, inequalities $(22)_1$ and $(23)_1$ imply inequality $(24)_1$. In conclusion, inequalities (22) and (23) imply inequalities (24). Thus, f is *-pseudomonotone decreasing. Consider a point

$$\begin{aligned} x^0 &= \left(x_1^0, x_2^0, x_3^0\right)^\top \in f^{-1}\left(K^*\right) \\ &= \left\{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + x_3 \in [0, 1000],\ x_1 + x_2 \in [0, 1000],\ x_1 \in [0, 10]\right\}. \end{aligned}$$

Of course, there are infinitely many such points. For example, it is easy to see that the box $[0, 10] \times [0, 490] \times [0, 500]$ is contained in $f^{-1}(K^*)$, so one could choose x^0 from this box. Thus, if we consider the recursion (18) with f defined by (21) and starting from x^0, then, by Theorem 5, the sequence $\{x^n\}_{n\in\mathbb{N}}$ defined by this recursion is convergent to a solution of the complementarity problem defined by f and the cone K (given by (20)).

The next theorem gives a sufficient condition for the recursion (18) to be convergent to a nonzero solution.

Theorem 6. *Let H be a Hilbert space, $K \subset H$ a regular *-isotone projection cone and $f : K \to H$ a *-pseudomonotone decreasing, continuous mapping such that $f^{-1}(K^*) \neq \emptyset$. Let $J : K \to H$ be the inclusion mapping defined by $J(x) = x$ and $P_K : H \to K$ the projection mapping onto K. If there are $\hat{x} \in f^{-1}(K^*)$ and $u \in \hat{x} + K$ such that*

$$\left(P_K \circ (J - f)\right)\left((\hat{x} + K) \cap (u - K) \cap f^{-1}\left(K^*\right)\right) \subset \hat{x} + K,$$

then $\hat{x}$ is a solution of the nonlinear complementarity problem NCP(f, K) and for any $x^0 \in (\hat{x} + K) \cap (u - K) \cap f^{-1}(K^*)$ the recursion (18) *starting from x^0 is convergent and its limit x^* is a solution of the nonlinear complementarity problem NCP(f, K) such that $\hat{x} \leqslant_K x^* \leqslant_K u$. In particular, if $\hat{x} \neq 0$, then the recursion* (18) *is convergent to a nonzero solution.*

Proof. Since, $\hat{x} \in (\hat{x} + K) \cap (u - K) \cap f^{-1}(K^*)$ and

$$\left(P_K \circ (J - f)\right)\left((\hat{x} + K) \cap (u - K) \cap f^{-1}\left(K^*\right)\right) \subset \hat{x} + K,$$

we have $\hat{x} \leqslant_K (P_K \circ (J - f))(\hat{x}) = P_K(\hat{x} - f(\hat{x})) \leqslant_K \hat{x}$. Hence,

$$\hat{x} = P_K\left(\hat{x} - f(\hat{x})\right),$$

that is, $\hat{x}$ is a solution of the nonlinear complementarity problem *NCP*(f, K). In the proof of Theorem 5 we have seen by induction that

$$x^n \in K \cap f^{-1}\left(K^*\right), \tag{25}$$

for all $n \in \mathbb{N}$. We prove by induction the proposition

$$(\Omega_n) \quad \hat{x} \leqslant_K x^n \leqslant_K u, \tag{26}$$

for all $n \in \mathbb{N}$. Obviously, (Ω_0) is true. Suppose that (Ω_n) is true. Hence, by using relation (25), we have $x^n \in (\hat{x} + K) \cap (u - K) \cap f^{-1}(K^*)$. Thus,

$$\begin{aligned} x^{n+1} &= \left(P_K \circ (J - f)\right)\left(x^n\right) \\ &\in \left(P_K \circ (J - f)\right)\left((\hat{x} + K) \cap (u - K) \cap f^{-1}\left(K^*\right)\right) \subset \hat{x} + K. \end{aligned} \tag{27}$$

On the other hand, by using relation (25) and the (K^*, K)-isotonicity of P_K, we have

$$x^{n+1} = P_K\left(x^n - f\left(x^n\right)\right) \leqslant_K P_K\left(x^n\right) = x^n \leqslant_K u. \tag{28}$$

Relations (27) and (28) imply that (Ω_{n+1}) is also true. Taking the limit in relation (26), as n tends to infinity, we get $\hat{x} \leqslant_K x^* \leqslant_K u$. $\square$

Definition 5. Let H be a Hilbert space, $K \subset H$ a cone, $f : K \to H$ a mapping and $L > 0$. The mapping f is called **-order weekly L-Lipschitz* if

$$f(x) - f(y) \leqslant_{K^*} L(x - y),$$

for all $x, y \in K$ with $y \leqslant_K x$. If $L = 1$, then f is called **-order weekly nonexpansive*.

It is easy to see that the mapping f is *-order weekly L-Lipschitz if and only if the mapping $K \ni x \mapsto Lx - f(x)$ is *-increasing.

Definition 6. Let H be a Hilbert space, $K \subset H$ a cone, $f : K \to H$ a mapping and $L > 0$. Then, the mapping f is called *projection order weekly L-Lipschitz* if the mapping $K \ni x \mapsto P_K(Lx - f(x))$ is K-isotone where P_K is the projection mapping onto K. If $L = 1$ the mapping f is called *projection order weekly nonexpansive*.

If $K \subset H$ is a *-isotone projection cone, then it is easy to see that every *-order weekly L-Lipschitz mapping is projection order weekly L-Lipschitz. In particular, every *-order weekly nonexpansive mapping is projection order weekly nonexpansive. Therefore, the next theorem is also true if we replace the projection order weekly L-Lipschitz condition for f with the *-order weekly L-Lipschitz condition.

Theorem 7. *Let H be a Hilbert space, $K \subset H$ a regular *-isotone projection cone, $L > 0$ and $f : K \to H$ a *-pseudomonotone decreasing, projection order weekly L-Lipschitz, continuous mapping such that*

$$f^{-1}\left(K^*\right) \neq \emptyset.$$

Let $\hat{x}$ be a solution of the nonlinear complementarity problem NCP(f, K). Then, for any $x^0 \in (\hat{x} + K) \cap f^{-1}(K^*)$ the recursion

$$x^{n+1} = P_K\left(x^n - \frac{f(x^n)}{L}\right) \tag{29}$$

starting from x^0 is convergent and its limit x^* is a solution of the nonlinear complementarity problem NCP(f, K) such that $\hat{x} \leqslant_K x^*$. In particular, if $\hat{x} \neq 0$, then the recursion (29) is convergent to a nonzero solution.

Proof. We will use the following well-known property of the projection mapping P_κ onto a cone κ: $P_\kappa(\lambda x) = \lambda P_\kappa(x)$ for all $x \in H$ and $\lambda > 0$. We remark that the nonlinear complementarity problem $NCP(f, K)$ is equivalent to the nonlinear complementarity problem $NCP(f/L, K)$. Denote $g = f/L$. Then, the recursion (29) can be written in the form

$$x^{n+1} = P_K\big(x^n - g\big(x^n\big)\big).$$

We will use Theorem 6 for the mapping g. Let $J : K \to H$ be the inclusion mapping defined by $J(x) = x$ and $u \in x^0 + K$ arbitrary. Since any solution of the nonlinear complementarity problem $NCP(g, K)$ is a solution of the nonlinear complementarity problem $NCP(f, K)$ too, it is enough to check the relation

$$\big(P_K \circ (J - g)\big)\big((\hat{x} + K) \cap (u - K) \cap g^{-1}\big(K^*\big)\big) \subset \hat{x} + K. \tag{30}$$

We have

$$P_K\big(x - g(x)\big) = P_K\left(\frac{1}{L}\big(Lx - f(x)\big)\right) = \frac{1}{L} P_K\big(Lx - f(x)\big), \tag{31}$$

for all $x \in K$. Since the mapping f is projection order weekly L-Lipschitz, from relation (31) and the scale invariance of the ordering induced by K, it follows that the mapping g is projection order weekly nonexpansive. For each $x \in (\hat{x} + K) \cap (u - K) \cap g^{-1}(K^*)$ we have $\hat{x} \leqslant_K x$. Thus, since $K \ni x \mapsto P_K(x - g(x))$ is K-isotone and $\hat{x}$ is a solution of the nonlinear complementarity problem $NCP(g, K)$, it follows that

$$\hat{x} = P_K\big(\hat{x} - g(\hat{x})\big) \leqslant_K P_K\big(x - g(x)\big).$$

The previous relation can be rewritten as $(P_K \circ (J - g))(x) \in \hat{x} + K$. Therefore, relation (30) holds. □

Example 2. We will use the notations from Example 1. Let $L > 0$ be a constant and $f : K \to \mathbb{R}^3$ a *-decreasing mapping. We will analyze under which conditions is the mapping $x \mapsto f(x) - Lx$ also *-decreasing. Let $E = (U^\top)^{-1}$. Then, $K = E\mathbb{R}^3_+$, $K^* = U\mathbb{R}^3_+$, $E = [e^1, e^2, e^3]$ and $U = [u^1, u^2, u^3]$. The column vectors e_1, e_2, e_3 are the generators of K and the column vectors u^1, u^2, u^3 are the generators of K^*. Any element $x \in K$ can be uniquely written as

$$x = x^e_1 e^1 + x^e_2 e^2 + x^e_3 e^3.$$

We also have the unique decomposition

$$f(x) = f^u_1(x)u^1 + f^u_2(x)u^2 + f^u_3(x)u^3.$$

Denote the components of x with respect to the canonical basis of $\mathbb{R}^3$ by x_1, x_2, x_3 and the components of $f(x)$ with respect to the canonical basis of $\mathbb{R}^3$ by $f_1(x)$, $f_2(x)$, $f_3(x)$. We will next use the terminology of a decreasing, increasing function in the classical sense. It is easy to see that f is *-decreasing if and only if f^u_1, f^u_2, f^u_3 are decreasing with respect to each variable x^e_1, x^e_2, x^e_3. In other words f is *-decreasing if and only if

$$f(x) = g_1(x)u^1 + g_2(x)u^2 + g_3(x)u^3, \tag{32}$$

where g_1, g_2, g_3 are decreasing with respect to each variable x^e_1, x^e_2, x^e_3. We have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = E \begin{pmatrix} x^e_1 \\ x^e_2 \\ x^e_3 \end{pmatrix},$$

from where we get

$$\begin{pmatrix} x^e_1 \\ x^e_2 \\ x^e_3 \end{pmatrix} = U^\top \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

or equivalently

$$\begin{cases} x^e_1 = x_1, \\ x^e_2 = x_1 + x_2, \\ x^e_3 = x_1 + x_2 + x_3. \end{cases} \tag{33}$$

Let $f : K \to \mathbb{R}^3$ be a *-decreasing mapping. Then, by Eq. (32), there exist g_1, g_2, g_3 decreasing with respect to each variable x^e_1, x^e_2, x^e_3 such that

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = U \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

or equivalently

$$\begin{cases} f_1 = g_1 + g_2 + g_3, \\ f_2 = g_2 + g_3, \\ f_3 = g_3. \end{cases} \tag{34}$$

Let $h : K \to \mathbb{R}^3$ be defined by $h(x) = f(x) - Lx$, where $L > 0$ and h_1, h_2, h_3 the components of $h(x)$ with respect to the canonical basis and h_1^u, h_2^u, h_3^u the components of $h(x)$ with respect to the basis (u^1, u^2, u^3). We have to analyze when h_1^u, h_2^u, h_3^u are decreasing with respect to each variable x_1^e, x_2^e, x_3^e. From Eqs. (34) we get

$$\begin{cases} h_1 = g_1 + g_2 + g_3 - Lx_1, \\ h_2 = g_2 + g_3 - Lx_2, \\ h_3 = g_3 - Lx_3. \end{cases} \tag{35}$$

We have

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = U \begin{pmatrix} h_1^u \\ h_2^u \\ h_3^u \end{pmatrix},$$

from which we get

$$\begin{pmatrix} h_1^u \\ h_2^u \\ h_3^u \end{pmatrix} = E^\top \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},$$

or equivalently

$$\begin{cases} h_1^u = h_1 - h_2, \\ h_2^u = h_2 - h_3, \\ h_3^u = h_3. \end{cases} \tag{36}$$

Combining Eqs. (35) and (36) we get

$$\begin{cases} h_1^u = g_1 - L(x_1 - x_2), \\ h_2^u = g_2 - L(x_2 - x_3), \\ h_3^u = g_3 - Lx_3, \end{cases} \tag{37}$$

or, by using Eqs. (33)

$$\begin{cases} h_1^u = g_1 - L\big(2x_1^e - x_2^e\big), \\ h_2^u = g_2 - L\big(2x_2^e - x_1^e - x_3^e\big), \\ h_3^u = g_3 - L\big(x_3^e - x_2^e\big). \end{cases} \tag{38}$$

Since g_3 is decreasing with respect to each variable x_1^e, x_2^e and x_3^e, Eq. $(38)_3$ implies that h_3^u is decreasing with respect to x_1^e, x_3^e. Eq. $(38)_1$ implies that h_1^u is decreasing with respect to x_1^e, x_3^e. Eq. $(38)_2$ implies that h_2^u is decreasing with respect to x_2^e. It follows that $x \mapsto f(x) - Lx$ is *-decreasing if and only if h_1^u, h_3^u are decreasing with respect to x_2^e and h_2^u is decreasing with respect to x_1^e, x_3^e. Thus, $x \mapsto f(x) - Lx$ is *-decreasing if and only if $g_1 = k_1 - Lx_2^e$, $g_2 = k_2 - Lx_1^e - Lx_3^e$ and $g_3 = k_3 - Lx_2^e$, where $k_1 : K \to \mathbb{R}$, $k_2 : K \to \mathbb{R}$ and $k_3 : K \to \mathbb{R}$ are decreasing functions with respect to each variable x_1^e, x_2^e, x_3^e.

Thus, if we consider a mapping $f : K \to \mathbb{R}^3$ defined by Eq. (32) with $g_1 = k_1 - Lx_2^e$, $g_2 = k_2 - Lx_1^e - Lx_3^e$ and $g_3 = k_3 - Lx_2^e$, where $k_1 : K \to \mathbb{R}$, $k_2 : K \to \mathbb{R}$, $k_3 : K \to \mathbb{R}$ are decreasing functions with respect to each variable x_1^e, x_2^e, x_3^e, then f is *-order weekly L-Lipschitz and therefore projection order weekly L-Lipschitz. Moreover, f is *-decreasing and therefore *-pseudomonotone decreasing. Thus, we can use Theorem 7 to conclude that the nonlinear complementarity problem $NCP(f, K)$ has a solution $\hat{x}$ and the recursion (18) starting from any $x^0 \in (\hat{x} + K) \cap f^{-1}(K^*)$ converges to a solution x^* of $NCP(f, K)$ such that $\hat{x} \leqslant_K x^*$.

6. Conclusions

In this paper we analyzed the convergence of the recursion

$$x^{n+1} = P_K\big(x^n - f\big(x^n\big)\big),$$

where K is a closed convex cone in a Hilbert space H and $f : K \to H$ is a continuous mapping. If the sequence $\{x^n\}_{n\in\mathbb{N}}$ is convergent to $x^* \in K$, then x^* is a solution of the nonlinear complementarity problem defined by K and f. In [4] a set of sufficient conditions is given for the convergence of $\{x^n\}_{n\in\mathbb{N}}$. This paper presented another set of sufficient conditions for the convergence of $\{x^n\}_{n\in\mathbb{N}}$ for a different class of cones introduced by us, the so-called *-isotone projection cones. We also considered the above recursion with f replaced by a positive scaling of f.

As far as we know the notions of *-isotone projection cones, *-pseudomonotone decreasing mappings, *-order weekly L-Lipschitz and *-order weekly nonexpansive were first considered in this paper. In the future we plan to find classes of projection order weekly L-Lipschitz mappings that are not order weekly L-Lipschitz, to extend the notion of *-pseudomonotone decreasing mappings to the notion of *-projection pseudomonotone decreasing mappings and to give classes of *-projection pseudomonotone decreasing mappings that are not *-pseudomonotone decreasing mappings. Accomplishing these goals would lead to more general results on more general cones. We also plan to analyze the extragradient iteration of the type $x^{n+1} = P_K(x^n - \lambda f(P_K(x^n - \lambda f(x^n))))$ considered in [19–22].

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